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S-N. CHOW

VIRTUAL PERIODS AND GLOBAL CONTINUATION OF PERIODIC ORBITS

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# Virtual periods and global continuation of periodic orbits *)

by

S-N. Chow **)

ABSTRACT

In this paper we improve an earlier result about global bifurcation of periodic orbits under the restriction that the phase space is three or four dimensional.

KEY WORDS & PHRASES: *ordinary differential equation, Fuller index, virtual period, Hopf bifurcation, global bifurcation*

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**) Department of Mathematics Michigan State University
East Lansing, Michigan 48824 USA

Consider an autonomous ordinary differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1 \tag{1}$$

and let $x(t,a)$ denote the solution with initial point $a \in \mathbb{R}^n$. Let $\Omega \subseteq (0,\infty) \times \mathbb{R}^n$ be an open bounded set, bounded away from $\{0\} \times \mathbb{R}^n$, whose boundary $\partial\Omega$ is free of periodic solution of (1). That is

$$x(t,a) \neq a \quad \text{for all } (t,a) \in \partial\Omega \ .$$

Note that critical points are considered as periodic points with arbitrary period, hence $\Omega$ contains no such points. The Fuller degree $d(\Omega,f)$ is a rational number defined for the flow (1) and the set $\Omega$. Moreover, if $f^\alpha$, $0 \le \alpha \le 1$, is a homotopy of vector fields then $d(\Omega,f^\alpha)$ is independent of $\alpha$ provided $\partial\Omega$ is free of periodic solution of $\dot{x} = f^\alpha(x)$. In particular $d(\Omega,f) = d(\Omega,g)$ for $g$ uniformly near $f$, so to give a computational formula for the degree it is sufficient to consider the generic case when all periodic solutions of (1) are hyperbolic.

In this generic case $\Omega$ contains only finitely many periodic orbits and the Fuller degree is defined as

$$d(\Omega,f) = \sum i(\Gamma) \tag{2}$$

where $\Gamma \subseteq \Omega$ is a periodic orbit and $i(\Gamma)$ is a rational number, the Fuller index. To define $i(\Gamma)$, let $\gamma \subseteq \mathbb{R}^n$ be a non-constant periodic orbit, say

$$\gamma = \{x(t,a) \mid 0 \le t \le T\}$$

with least period

$$T = \inf\{t > 0 \mid x(t,a) = a\} \in (0,\infty).$$

If k is a positive integer, set

$$\Gamma = \{kT\} \times \gamma \subseteq (0,\infty) \times \mathbb{R}^n$$

and define

$$i(\Gamma) = (1/k)(-1)^\sigma \tag{3}$$

where $\sigma$ is the number of eigenvalues of $(\partial x/\partial a)(kT,a)$ in the interval $(1,\infty)$. Here $\sigma$ depends on the parity of k and $i(\Gamma)$ depends on k itself. It important to note the summation (2) is taken over precisely those k and $\gamma$ for which $\Gamma \subseteq \Omega$. Formula (3) refers to the generic case when all periodic solutions of (1) are hyperbolic. Observe in particular that

$$i(\Gamma) = (1/k)\mathrm{ind}(\pi^k) \tag{4}$$

where ind denotes the fixed point index and $\pi^k$ is the kth iterate of the Poincaré map for y. Now for arbitrary f suppose that some orbit $\gamma$, though not necessarily hyperbolic, is isolated from periodic orbits with period near kT, for some given k. It thus corresponds to an isolated fixed point of $\pi^k$, so that $i(\Gamma)$ may be defined by (4). Also, because $\Gamma$ is isolated in $(0,\infty) \times \mathbb{R}^n$ the quantity $d(\Omega_0,f)$ is defined for small enough neighborhoods $\Omega_0$ of $\Gamma$. It is a fact, not difficult to show, that $i(\Gamma) = d(\Omega_0,f)$. This means that for any f, as long as $\Omega$ contains only finitely many orbits $\Gamma$, the Fuller degree $d(\Omega,f)$ may be calculated from (2), (4).

For more information, we refer the reader to [1], [2] and [3]. By using Fuller degree, one may prove a global version of the Hopf bifurcation theorem which was first proved in [4]. Related results may be found in [5], [6] and [7]. In the following, we state the theorem (Theorem 1) which was shown by using Fuller degree [7].

Consider a parametrized differential equation

$$\begin{aligned} &\dot{x} = f(x,\alpha) \\ &f : \mathbb{R}^n \times R \to \mathbb{R}^n \text{ smooth} \\ &f(x,\alpha) = A(\alpha)x + O(|x|^2) \quad \text{near } x = 0 \end{aligned} \tag{5}$$

as described above. Assume the $n \times n$ matrix $A(\alpha)$ is nonsingular for all $\alpha$, and let $P \subseteq \mathbb{R}$ denote the set of values $\alpha$ for which $A(\alpha)$ has an eigenvalue on the imaginary axis.

Fix any isolated point $\alpha_0 \in P$ and let $\pm i\omega_0$, $\omega_0 > 0$, be a pair of eigenvalues of $A(\alpha_0)$. As the integer multiples of $i\omega_0$ may also be eigenvalues of $A(\alpha_0)$, let

$m(c)$ = the generalized (or algebraic) multiplicity of $ic\omega_0$ as an eigenvalue of $A(\alpha_0)$, for $c = 1,2,3,\ldots$ .

Thus $m(1) > 0$ and $m(c) = 0$ for large c. For $|\alpha - \alpha_0| \neq 0$ small there are no eigenvalues of $A(\alpha)$ on the imaginary axis near any $ic\omega_0$. Hence $ic\omega_0$ splits into various eigenvalues nearby, some in the left half plane and some in the right, but still with total multiplicity $m(c)$. For small $\varepsilon > 0$ let

$r^{\pm}(c)$ = the generalized multiplicity of those eigenvalues of $A(\alpha)$, near $ic\omega_0$, which are in the right half plane, for $0 < \pm(\alpha-\alpha_0) < \varepsilon$.

Thus $0 \le r^{\pm}(c) \le m(c)$, and the corresponding multiplicity of eigenvalues in the left half plane is $m(c) - r^{\pm}(c)$. Finally, set

$$r(c) = r^{+}(c) - r^{-}(c).$$

To describe the bifurcation, let

$$\left.\begin{aligned} B &= \{(T,0,\alpha) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R} \mid \alpha \in P,\ T = 2\pi k/|\omega| \text{ where } i\omega \text{ is an eigenvalue of } A(\alpha) \text{ and } k > 0 \text{ is an integer}\},\\ \Lambda &= \{(T,a,\alpha) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}\ x(T,a,\alpha) = a\},\\ K &= (\Lambda - (0,\infty) \times \{0\} \times \mathbb{R}) \cup B, \end{aligned}\right\} \quad (6)$$

where $x(t,a,\alpha)$ denotes the solution of (5) with $x = a$ at $t = 0$. Since periodic solutions near $x = 0$ can only exist when $A(\alpha)$ has eigenvalues $\pm i\omega$ on the imaginary axis, and then only with periods near $2\pi k/|\omega|$, $k = 1,2,3,\ldots$, it follows that B represents the possible bifurcation points of periodic solutions from $x = 0$. Consider the values $\alpha_0$ and $\omega_0$ chosen above and for $c = 1,2,\ldots$ let

$$p_c = (2\pi/(c\omega_0),0,\alpha_0)$$

and

$K_c$ = the maximal connected component of $\overline{K}$ containing $p_c$ ($K_c = \emptyset$ if $p_c \notin B$).

We may now state the main theorem.

THEOREM 1. *Assume*

$$\sum_{c=1}^{\infty} \frac{1}{c}\gamma(c) \neq 0. \qquad (7)$$

*Then either*

(1) $K_1$ *contains a point* $(T,a,\alpha) \neq (2\pi/\omega_0, 0, \alpha_0)$ *where* $T > 0$ *and* $(a,\alpha)$ *is a critical point of equation (5); or*

(2) $K_1$ *is disjoint from* $\{0\} \times \mathbb{R}^n \times \mathbb{R} \subseteq \Lambda$ *is unbounded in* $(0,\infty) \times \mathbb{R}^n \times \mathbb{R}$, *that is, contains points* $(T,a,\alpha)$ *with* $T + |a| + |\alpha|$ *arbitrarily large.*

Applications to functional differential equations may be found in [3]. In [5], Theorem 1 is used to prove Liapunov center theorem. However, even though Theorem 1 is "global" but it becomes a "local" result for Liapunov center theorem. The reason is that $K_1$ in Theorem 1 may be unbounded in $(0,\infty) \times \mathbb{R}^n \times \mathbb{R}$, but $\{(a,\alpha): (T,a,\alpha) \in K_1\}$ may be bounded together with the least period $T_0$ associated with the periodic orbit through the point $(a,\alpha)$. To illustrate this point, consider the following example in [8]. A parametrized differential equation

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^4, \quad 0 \leq \alpha \leq 1, \qquad (8)$$

is constructed in [8] with the following properties:

(1) equation (8) has an isolated periodic orbit $\gamma(\alpha)$ for all $0 \leq \alpha \leq 2/3$;

(2) $\gamma(\alpha)$, $0 \leq \alpha \leq 1$, is hyperbolic except at $\alpha = 1/3, 2/3$;

(3) $\gamma(1/3)$ has a generic period doubling bifurcation, i.e., a second family of periodic orbits $\gamma_1(\alpha)$, $1/3 \leq \alpha \leq 2/3$, bifurcates from $\alpha(1/3)$ and the least periods of $\gamma_1(\alpha)$ for $\alpha$ near 1/3 are approximatily twice that of $\gamma(1/3)$;

(4) $\gamma(2/3)$ has a generic saddle-node bifurcation, i.e., $\gamma_1(\alpha)$ and $\gamma(\alpha)$ coalesce and annihilate each other at $\alpha = 2/3$.

In Figure 1, a schematic diagram of this example is shown.

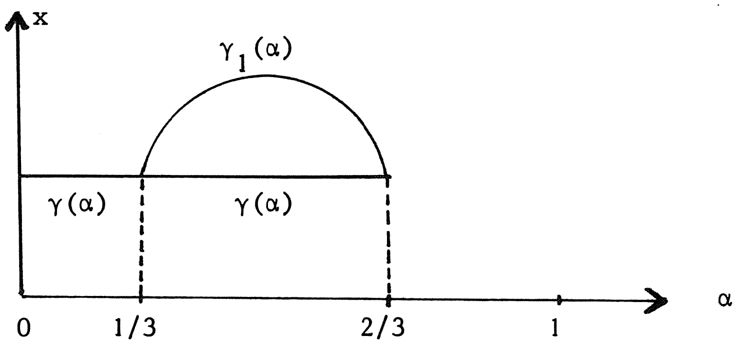


Fig. 1

If $\gamma(\alpha)$ and the periods of $\gamma(\alpha)$ were elements of $K_1$ in Theorem 1, then the set $K_1$ would be unbounded. This is shown in Figure 2, where the orbits are represented by the parameter $\alpha$ and their periods T and the least periods of $\gamma(\alpha)$ are assumed to be $T_0 > 0$. We note that

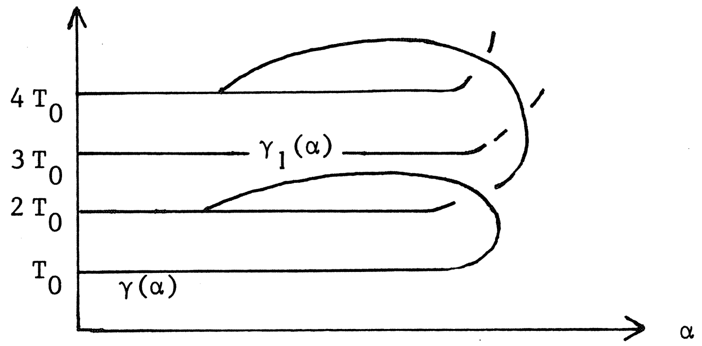


Fig. 2

the sit $\{(a,\alpha) = (T,a,\alpha) \in K_1\}$ with its least periods is bounded.
On the other hand, in order for $\gamma(\alpha)$, $0 \le \alpha < 1/3$, to have such behavior, $\gamma(\alpha)$ must have a non-orientable unstable manifold and such orbits could not be connected to a Hopf bifurcation point without any bifurcations ([9], [10]).
This indicates the possibility to extend Theorem 1 to include least periods.
In this report, we will present a theorem which says essentially Theorem 1 is true if we replace periods by least periods provided the phase space $\mathbb{R}^n$ is 3 or 4 dimensional, i.e., n = 3 or 4. This result is new and was found in collaburation with K. Alligood, J. Mallet-Paret and J. Yorke.
The following definition is essential in our approach.

DEFINITION 2. Let $\gamma$ be a periodic orbit of (1) with least period $T_0 > 0$ and $\pi$ be its Poincaré map at $a \in \mathbb{R}^n$. Let $A = D\pi(a)$ be the derivative of $\pi$ at a and

$$M = \{m \ge 1 : \text{there exists } x \in \mathbb{R}^{n-1} \text{ with } x, Ax, \ldots, A^{m-1}x \text{ distinct, but } x = A^m x\}.$$

We say T is a *virtual period* of $\gamma$ if $T = m\,T_0$ for some $m \in M$.

DEFINITION 3. $\gamma$ is said to be a *nice periodic orbit* of (1) if the Poincaré map $\pi$ of $\gamma$ at satisfies the condition that a is an isolated fixed point for each iterate $\pi^k$, k = 1,2,3,..., though the neighborhood of isolation may depend on k.
The following theorems indicate the role of virtual periods.

<u>THEOREM 4 [10].</u> *Let* $\gamma$ *be a nice periodic orbit and* $\pi$ *be the Poincaré map of* $\gamma$ *at* a. *Let* $\tilde{\pi}$ *be* $C'$*-close to* $\pi$. *Then a necessary condition for there to exists* b *close to* a *with* b, $\tilde{\pi}(b), \ldots, \tilde{\pi}^{m-1}(b)$ *distinct, but* $b = \tilde{\pi}^m(b)$ *is that* $m \in M$, *where* M *is as in Definition 2.*

<u>THEOREM 5 [10].</u> *Let* $\gamma$ *be a nice periodic orbit and* $\pi$ *be the Poincaré map of* $\gamma$ *at* $a \in \mathbb{R}^n$. *Let* $k_m$ *denote the fixed point index of* $\pi^m$, $m \geq 1$. *Then the vector* $k = (k_1, k_2, \ldots)$ *has the form*

$$k = \begin{cases} \sum\limits_{m \in M} c_m j_m & \sigma- = \text{even} \\ \sum\limits_{m \in Me} c_m j_m + \sum\limits_{m \in M_0} c_m (j_m - j_{2m}), & \sigma- = \text{odd} \end{cases}$$

*where* $\sigma-$ *is the number of eigenvalues of the derative* $D\pi(a) = A$, *counting multiplicity, in* $(-\infty, -1)$, M *is the set in Definition 2,* $c_m$ *are integers,* $Me = \{m: m \in M, m \text{ is even}\}$, $M_0 = M \backslash Me$, *and* $j_m$ *is the vector* $j_m = (j_{ma})_{a=1}^{\infty}$ *with*

$$j_{ma} = \begin{cases} m, & \text{if } m \text{ divides } a, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5 says that the following definition is well-defined.

<u>DEFINITION 6.</u> Let $\gamma$ be a nice periodic orbit of (1). The $\phi$ *index of* $\gamma$, $\phi(\gamma)$, is defined by

$$\phi(\gamma) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} k_m$$

where $k_m$ is the fixed point index of $\pi^m$, the m th iterate of the Poincaré map $\pi$. It is not difficult to see from Theorem 5 that the following are true.

<u>PROPOSITION 7.</u> *The* $\Phi$*-index of a nice periodic orbit is an integer.*

<u>PROPOSITION 8.</u> *If* $\gamma$ *is a nice periodic orbit with a non-orientable unstable manifold, then* $\phi(\gamma) = 0$.

We have the following "generalization" of Theorem 1 in terms of virtual periods.

<u>THEOREM 9 [11].</u> *Let* $\gamma_0$ *be a nice periodic orbit of (5) for* $\alpha = \alpha_0$. *If the* $\phi$*-index,* $\phi(\gamma_0)$, *is nonzero and if* $\Gamma$ *is the component of periodic orbits of (5) containing* $\gamma_0$, *then either of the following conditions hold:*

(a) $\Gamma - (\gamma_0 \times \{\alpha_0\})$ *is connected or*

(b) *each of the two components* $\Gamma_i$, $i = 1,2$, *satisfies one of the following:*

(1) $\Gamma_i$ *is unbounded in* $(x,\alpha)$*-space,*
(2) $\bar{\Gamma}_i$ *contains a center, i.e., a generalized Hopf bifurcation point;*
(3) *the virtual periods of orbits in* $\Gamma_i$ *are unbounded.*

REMARK 10. By Proposition 8, the $\phi$-index of the periodic orbit $\gamma(\alpha)$ in Figure 1, for any $0 \le \alpha < 1/3$, is zero. This shows that the assumption, $\phi(\gamma_0) \neq 0$, is necessary in Theorem 9.

We are now ready to state and prove our main result.

THEOREM 10. *If the phase space* $\mathbb{R}^n$ *is 3 or 4 dimensional, then under the hypotheses of Theorem 9 condition (b3) may be replaced with the following stronger condition*

*(b3') the least periods of orbits in* $\Gamma_i$ *are unbounded.*

LEMMA 11. *A periodic orbit* $\gamma$ *in* $\mathbb{R}^3$ *or* $\mathbb{R}^4$ *has at most one virtual period in addition to the least period* $T_0 > 0$ *of* $\gamma$.

PROOF. Let $\mu_1, \ldots, \mu_k$ denote characteristic multipliers of $\gamma$ (k=2 or 3). Note that $2T_0$ is a virtual period if and only if $\mu_i = -1$ for some i; $mT_0$, $m \ge 3$, is a virtual period if and only if for some $i \neq j$, $\mu_i = \bar{\mu}_j$, $\mu_i^m = 1$ but $\mu_i^p \neq 1$ for any $1 \le p < m$. If the phase space $\mathbb{R}^n$ is 3-dimensional, then there exists at most one virtual period since $k = 2$. If $\mathbb{R}^n$ is 4-dimensional and there are two distinct virtual periods in addition to the least period $T_0$, then we may assume $\mu_1 = -1$, $\mu_2 = e^{-i\theta}$, $\mu_3 = e^{i\theta}$. The product $\mu_1\mu_2\mu_3 = -1$. This contradicts that the Poincaré map is orientation preserving.

PROOF OF THEOREM 10. Suppose no other conditions in Theorem 9 are satisfied except (b3) for $\Gamma_i$. We will show that (b3') is satisfied by $\Gamma_i$.

It can be shown as in [11] that if $a_2 > a_1$, are sufficiently large, there exists a compact connected set $Q \in \Gamma_i$ such that $(\gamma,\alpha) \in Q$ implies the virtual period of $\gamma$ lies in $[a_2, 2a_1]$. Furthermore, for each $a \in [a_1, a_2]$, there exists $(\gamma,\alpha) \in Q$ such that the virtual period of $\gamma$ is in $[a, 2a)$.

Suppose (b3') is false. Then there exist $T_2 > T_1 > 0$ such that $(\gamma,\alpha) \in \Gamma_i$ implies the least period of $\gamma$ is in $[T_1,T_2]$. We may assume

$$a_1 > T_2, \quad a_2 > \frac{2T_2}{T_1} a_1 .$$

By Lemma 11, there is at most one virtual period for $\gamma$. Denote the least periods and virtual periods by $T_0(\gamma,\alpha)$ and $m(\gamma,\alpha)T_0(\gamma,\alpha)$ for $(\gamma,\alpha) \in Q$. By the property of Q, there exist $(\gamma_1,\alpha_1)$, $(\gamma_2,\alpha_2) \in Q$ such that $m(\gamma_j,\alpha_j)T_0(\gamma_j,\alpha_j) \in [a_j,2a_j)$, $j = 1,2$. This implies $m(\gamma_1,\alpha_1) < m(\gamma_2,\alpha_2)$. We will obtain a contradiction by showing $m(\gamma,\alpha)$ is constant for $(\gamma,\alpha) \in Q$.

Since Q is compact and connected, if suffices to show that $m(\gamma,\alpha)$ is continuous on Q. This amounts to showing the least period $T_0(\gamma,\alpha)$ is continuous on Q. If $(\gamma_1,\alpha_1) \in Q$, then $T_0(\gamma_1,\alpha_1)$ is near $T_0(\tilde{\gamma},\tilde{\alpha})$ or $m(\tilde{\gamma},\tilde{\alpha})T_0(\tilde{\gamma},\tilde{\alpha})$ for $(\tilde{\gamma},\tilde{\alpha})$ near $(\gamma_1,\alpha_1)$. But the latter is impossible, because $m(\tilde{\gamma},\tilde{\alpha})T_0(\tilde{\gamma},\tilde{\alpha}) \geq a_1 > T_2$, violationg the bounds on the least periods. Thus $T_0(\gamma,\alpha)$ is continuous on Q.

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